

# PRACTICE EXAM

## RANDOM GEOMETRY AND TOPOLOGY B

27 October 2023

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- You have from 15:00 until 17.00.
- It is not allowed to use phones, computers, books, notes or any other aids.
- Write each exercise on a separate sheet, with your name and student number clearly legible on each sheet.

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Consider the **biased voter model**  $(\xi_t)_{t \geq 0}$  with parameter  $\lambda > 1$  on the state space  $S = \{0, 1\}^{\mathbb{Z}}$ . This is a version of the voter model in which individuals are more likely to adopt the opinion 1. The process evolves according to the following transition rules. At time 0 every site is either in state 0 or 1. After an exponential time of parameter  $\lambda$  (that is with mean  $1/\lambda$ ) each site chooses one of its neighbors uniformly at random and flips to 1 if the chosen neighbor is in state 1. Furthermore, after an exponential time of parameter 1 each site chooses one of its neighbors uniformly at random and flips to 0 if that neighbor is in state 0.

### Exercise 1 (5 pts)

Write down the generator of the process. Is the process monotone? Why/why not?

### Exercise 2 (15 pts)

Describe a graphical representation of the process and define  $\xi_t$ , the state of the process at some time  $t > 0$ , in terms of this representation.

### Exercise 3 (10 pts)

Denote by  $(\xi_t^A)_{t \geq 0}$  the process started from the initial configuration

$$\xi_0^A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases} \quad \forall x \in \mathbb{Z}$$

for some  $A \subset \mathbb{Z}$ . Furthermore, denote by  $\mathbb{P}_\lambda$  the law of the process with parameter  $\lambda > 1$ . Prove that for any  $A$  finite

$$\mathbb{P}_\lambda(\xi_t^A \neq \underline{0} \ \forall t \geq 0) \text{ is non-decreasing in } \lambda.$$

### Exercise 4 (15 pts)

Define

$$\lambda_c := \inf\{\lambda : \mathbb{P}_\lambda(\xi_t^A \neq \underline{0} \ \forall t \geq 0) > 0\}.$$

Show that the value of  $\lambda_c$  does not depend on the choice of  $A$  (as long as it is finite).

### Exercise 5 (15 pts)

Prove that  $\lambda_c < \infty$ .

### Exercise 6 (10 pts)

Denote by  $\bar{\nu}_\lambda$  the upper invariant law of the process with parameter  $\lambda$ . Show that for all  $\lambda > 0$  and  $A \subset \mathbb{Z}$  finite

$$\mathbb{P}_\lambda(\xi_t^A \neq \underline{0} \ \forall t \geq 0) \neq \bar{\nu}\{\eta \in S : \eta(x) = 1 \ \forall x \in A\}.$$